Solution of Laguerre Equation by Frobenius Method

Deepti Bhatia[1], Ravneet Kaur[1*]

[1]*Department of Mathematics, Rayat & Bahra University, Sahauran, Mohali, India*

kaur.ravneet08@gmail.com

Received: 21 Dec 2024; **Accepted:** 5 Feb 2025; **Published:** 15 Feb 2025

Abstract:

The Frobenius method is a mathematical technique used to find power series solutions for linear differential equations with regular singular points. It involves assuming a power series solution, expressing coefficients in terms of recurrence relations, and determining convergence conditions. This method is particularly useful when traditional methods fails due to singular points of the differential equation. In the present work, Laguerre equation is solved using Frobenius method.

Keywords: Laguerre equation, Frobenius method, second-order linear equation, power series

1.Introduction

The Frobenius method plays a crucial role in solving problems in physics, engineering, and other fields where differential equations arise. It is a second-order linear differential equation that arises in various scientific and engineering contexts, quantum mechanics and mathematical physics. The significance of this equation is describing the radial part of the wave function for a hydrogen-like atom. The Laguerre equation involves a dependent variable, its derivatives, and a radial coordinate. Understanding and solving the Laguerre equation is essential for analyzing the behaviour of physical systems with radial symmetry which makes this equation an attractive equation for researchers.

Cite the manuscript as:
Deepti Bhatia and Ravneet Kaur (2025). Solution of Laguerre Equation by Frobenius Method, International Journal of Mathematics & Computational Frontiers, 1(1), pages 35-42.

For centuries, ordinary differential equations have been used in mathematics and remain one of the primary tools that are used for the modelization of phenomena [1] within all the sciences, engineering, and even economies. Most differential equations may be solved by using well-established methods, but when singularities are introduced in an equation, the system for solving becomes complicated. Singularities introduced into the differential equations further complicate the behaviour of the solutions, which demands specialized techniques to deal with them in an efficient manner. One of those methods, in fact powerful, is the Frobenius method, traditionally used for finding solutions near regular singular points.

In the classical application of the Frobenius method [2] is specially well suited for second order linear differential equations with regular singular points. In this technique if a point will be treated as a number from the given field, is a singular point then that point will be called regular provided a solution can be expanded at that point in a power series multiplied by a leading factor that accounts for the singularity. This gives the Frobenius method a series solution with point-wise convergence around the singular point, an important property when finding exact solutions is very difficult or even impossible to achieve. With this method, successful applications in a broad range of problems from mechanical systems as simple as they can be to complex equations like quantum mechanics and electromagnetic theory are possible.

A vast number of researchers contributed to the development of this extension of the Frobenius method beyond its classical application to regular singular points. In the literature, Einar Hille [3] indicated the possibility of applying this method to equations with irregular singular points, where the complexity of solutions drastically increases. Moreover, how solutions have asymptotic behaviour when approaching irregular singularities. It was analyzed that under certain conditions, the Frobenius method could be modified to obtain efficient results. This encouraged and generated new avenues in solving a larger class of differential equations arisen in pure and applied mathematics.

Levi-Civita took Hille's work further by refining techniques for dealing with irregular singular points. Levi-Civita's method has expanded the scope of classical Frobenius by moving it into such situations where, due to singularity, it is impossible to reach the behavior using power series. It was absolutely essential further applications in more difficult differential equations;

results of his work have been quoted as reference for mathematical physics, in which these differential equations emerge.

Around the same time, Rolf Nevanlinna [4] and Jean Écalle [5] further improved knowledge of the behavior of solutions at irregular singular points. Nevanlinna's worked on the analytic continuation that simplifies the solutions behaviour as they are continued past the immediate region of the singularities. Such a concept is important for solving ODEs with irregular singular points because of the complicated multivalued behavior often seen by the solutions, now requiring advanced techniques for continuation.

Another fundamental theory developed toward the end of the 20th century was that of Jean Écalle on resurgence, which became a helpful tool in handling such problems. This work was especially powerful in extending the Frobenius method and similar techniques to nonlinear differential equations and into the regions where irregular singular points are more commonly encountered.

Legendre, Bessel, and Hermite equations, which describe critical phenomena in fields from quantum mechanics to electromagnetism can also be analyzed by the present method. Thus, the Legendre equation arises in problems that have spherical symmetry, which contain the gravitational field and the electric potential of a charged sphere. The Bessel equation is important in speaking of wave propagation, and the Hermite equation becomes essential for quantum mechanics and more specifically for discussing the harmonic oscillator from a quantum point of view.

Despite these advances, there are many important classes of equations that have yet to be fully explored using the Frobenius method. Among them is the class of Laguerre equations, which also arises very naturally in quantum mechanics in the study of the Schrödinger equation for the hydrogen atom and many other phenomenon.

The sections of the present papers are: In section 2, the Laguerre equation is discussed in detail. In Section 3, the Frobenius method is elaborated. In Section 4 implementation of the method on the considered problem. Sections 5, presents the discussion and the conclusion is given in section 6.

2. Laguerre Equation

The Laguerre differential equation is a second-order linear equation which, having various many applications in physics and engineering. The Laguerre equation has the following form:

$$xy'' + (1-x)y' + ny = 0 \tag{1}$$

In equation (1), y is the solution and n is the nonnegative integer. The above equation is a special case of the more generalized Laguerre differential equation given as follow:

$$xy'' + (m+1-x)y' + (n-m)y = 0$$

The solutions of this equation are called the Laguerre polynomials and have found a very interesting application in many branches of theoretical physics, quantum mechanics for expansion of wave functions where the radial part is hydrogen atom [6]. Laguerre-type linear ordinary differential equations [7] are prototypes of structured linear differential equations of higher even-order, which naturally extend the second-order Bessel and Laguerre equations defined on the positive half-line of the real field R. This attracts the researchers to compute more relevant solution of such arising differential equations.

3. Frobenius Method

The power series method is very effective method for computing the solution of the ordinary differential equations where the equation contains variable coefficients. The general form of the power series is given as [12]:

$$\sum_{r=0}^{\infty} a_r (t-t_0)^r = a_0 + a_1(t-t_0) + a_2(t-t_0)^2 + ... \tag{2}$$

where 't' is varying, t_0 is the center of the series and $a_0, a_1, a_2, ...$ are the constant parameters.

Further if the equation is of the form:

$$t^2 y'' + tP(t)y' + Q(t)y = 0 \tag{3}$$

then reducing the equation in the standard form gives:

$$y'' + \frac{P(t)}{t^2} y' + \frac{Q(t)}{t} y = 0 \tag{4}$$

In such cases, the coefficients of y' and y are not analytical at $t=0$ which interprets that is have singularity at $t=0$. This insists Frobenius method to compute the solution as power series expanded form.

The generalized form of the solution is given as:

$$y=\sum_{k=0}^{\infty} a_k x^{r+k} \tag{5}$$

4. Implementation of Frobenius method on Laguerre differential equation

Frobenius method is applied to second order Laguerre differential equation by taking into special consideration the behaviour of the equation near its singular points. Singularity for the Laguerre equation occurs at $x=0$ and at $x=\infty$, where the former is a regular singular point. Applying the Frobenius method, a series solutions is obtained near point $x=0$ and determine under what conditions such solutions converge. Their behavior is also analyzed as $x\to 0$ to conclude whether they have an irregular singularity.

This work is derived from the basic research implemented by preceding mathematicians who enlarged the use of the Frobenius method with solutions to ever more complex problems. This work apparently presents the Laguerre equation, showing that the Frobenius method is extremely potent in solving differential equations with both theoretical and practical importance. Findings here would deepen our knowledge of the Laguerre equation itself but will give one insight about how series solutions are actually used for ODEs at singular points.

This extra technical purpose of this work would be to consider the practical consequences of these results to problems in quantum mechanics and other areas where the Laguerre polynomials are fundamental. It would also be interesting to apply Frobenius method for the Laguerre equation to relate new insights into the convergence properties of the series solutions together with some limitations of the method applied to equations showing complex singular behaviour. Such results can be good working tools not only for mathematicians but for physicists and engineers who will have a fresh opportunity to solve differential equations in their theoretical and practical uses.

Consider the equation:

$$xy'' + (1-x)y' + my = 0 \tag{6}$$

The general form of the equation is:

$$y'' + P(x)y' + \mathrm{Q(x)}y = 0 \tag{7}$$

It is observed that at $x_0 = 0,\ P(x) = 0$ is a regular singular point. It is assumed that the solution is:

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

Computing first and second order derivatives as:

$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1} \tag{8}$$

Similarly, $$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2} \tag{9}$$

Substituting the values in equation (6) which obtain the relation as:

$$x\left(\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-2}\right) + (1-x)\left(\sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1}\right) + m\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0 \tag{10}$$

$$\left(\sum_{n=0}^{\infty} a_n (n+r)(n+r-1)x^{n+r-1}\right) + \sum_{n=0}^{\infty} a_n (n+r)x^{n+r-1} - \left(\sum_{n=0}^{\infty} a_n (n+r)x^{n+r}\right) + m\left(\sum_{n=0}^{\infty} a_n x^{n+r}\right) = 0$$

$$\left(\sum_{n=0}^{\infty} a_n (n+r)^2 x^{n+r-1}\right) + \left(\sum_{n=0}^{\infty} a_n (m-n-r)x^{n+r}\right) = 0$$

Equating the coefficient of the lowest power of $x = 0$ on taking $n = 0$ we obtain:

$a_0 (r)^2 = 0$ which implies $r = 0$.

Continuing and equating x^{n+r-1} to zero by taking $n = k+1$ and similarly taking $n = k$ for x^{n+r} one obtains the relation as:

$$a_{k+1} = \frac{(k-m)}{(k+1)^2} a_k \text{ , as } r = 0$$

For $k = 0, 1, 2, \ldots$, the relations obtained are:

$$a_1 = \frac{-m}{(1)^2} a_0$$

$$a_2 = \frac{m(m-1)}{1^2.2^2} a_0$$

.

.

.

$$a_k = \frac{(-1)^k m!}{(k!)^2.(m-k)!} a_0$$

Required solution for $a_0 = 1$:

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k m!}{(k!)^2.(m-k)!} x^k$$

5. Discussion

It has been observed that the computed solution has varied applications in solving higher order differential equations. It signifies the importance of Laguerre polynomial in obtaining the solution of phenomenon in physics and hydrogen atom.

6. Conclusions

The implementation of the Frobenius method to the Laguerre equation, the analysis of its solutions is extended far beyond the one that could be obtained directly with a standard approach to power series and, in particular, gives insights into the Laguerre function near its singular points. It is exactly the adoption of a generalised power series expansion that could be inclusive of fractional or logarithmic terms that allows the Frobenius method to demonstrate solutions that do indeed capture the singular behaviour of the equation-precisely, at infinity. It is concluded that this method provide a better and extended solution. The future scope of this work is its extension over associated Laguerre equation.

Conflict of Interest

The authors have no conflict of interest.

Acknowledgments

I thank all the reviewers for their valuable suggestions. There was no dedicated grant for this study from any public, commercial, or non-profit organization.

References

[1] Abramowitz, M., & Stegun, I. A. (1972). *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. U.S. Department of Commerce, National Bureau of Standards.

[2] Arfken, G. B., & Weber, H. J. (2005). *Mathematical methods for physicists* (6th ed.). Academic Press.

[3] Hille, E. (1956). *Ordinary differential equations in the complex domain*. John Wiley & Sons.

[4] Nevanlinna, R. (1953). *Analytic functions*. Springer.

[5] Écalle, J. (1981). *Les fonctions résurgentes*. Hermann.

[6] Rabei, E., Al-Jamel, A., & Al-Masaeed, M. (2021). The solution of conformable Laguerre differential equation using conformable Laplace transform. *arXiv preprint arXiv:2112.01322*.

[7] Everitt, W. N., & Markett, C. (2011). Solution bases for the fourth-order Bessel-type and Laguerre-type differential equations. *Applicable Analysis*, *90*(3-4), 515-531.