International Journal of Mathematics & Computational Frontiers, Vol-1, Issue-01, pages 54-64, Jan-June 2025
https://www.ijmcf.com/

Comparison of RBF-PS Method and EMCB-DQ Method with Numerical Simulation of Fitzhugh-Nagumo Equation using LOOCV Approach

[1] Kiran Bala, [2] Richa Rani*
[1] Department of Mathematics, Govt. National College, Sirsa, Haryana
[2]Department of Mathematics, Guru Gorakhnath Ji Govt. College, Hisar, Haryana
kiranbalak21@gmail.com, richasardana7@gmail.com

Received:20 Feb 2025**; Accepted:** 15 Mar 2025**; Published:** 24 Mar 2025

Abstract

Numerical treatment of various PDEs is todays demanding area of research. In this article one of the PDEs has been simulated through two different numerical methods named mesh free the Radial basis Pseudo-spectral method (RB-PSM) and Exponential modified cubic B-Spline differential quadrature method (EMCB-DQM). Numerical results are derived using Leave-One-Out Cross-Validation (LOOCV) approach. This statistical approach is based on the concept of enhancing the positive characteristics of a mathematical model with diminishing negative aspects. Two examples of Fitzhugh-Nagumo equation represent the proposed numerical schemes by the obtained results LOOCV. Also, comparison of obtained results is also done for testing the accuracy and effectiveness of the presented different numerical methods.

Keywords: Exponential Cubic Differential quadrature method, Fitzhugh-Nagumo Equation, Radial Basis Pseudo Spectral Method, Leave-One-Out Cross-Validation approach.

1. Introduction

In mathematical research, non-linear PDEs are intensively practiced to emulate the physical phenomena of the natural model. Numerical solutions of PDEs provide valuable

Cite the manuscript as:
Kiran Bala, Richa Rani, (2025). Comparison of RBF-PS Method and EMCB-DQ Method, *International Journal of Mathematics & Computational Frontiers*, 1(1), pages 54-64.

facts and an improved form of mathematical models. The Fitzhugh-Nagumo (FN) equation is a type of non-linear partial differential equation presented as:

$$u_t = u_{xx} + u(u-\alpha)(1-u), \quad 0 < \zeta < 1 \tag{1}$$

Here, $u(x,t)$ is an anonymous factor depends upon the arbitrary constant. Nagumo and Fitzhugh proposed the FN equation for solving the problematic models of nerve axon and membrane [1-2]. The presented equation has various real-life applications such as in population genetics, autocatalytic chemical reactions, circuit theory, and neurophysiology [3-5].

In this paper, the Fitzhugh-Nagumo equation is solved numerically using the hybrid approach of one of the algorithms named LOOCV with two different basis function: exponential modified cubic B-spline and RBFs. The associated shape parameter of RBFs is found out by the proposed approach of LOOCV with one of the RBF method named RBF pseudo-spectral method and other named as exponential modified cubic B-Spline differential quadrature method with minimizing the errors.

1.1. Method 1:Exponential Modified Cubic Differential quadrature (EMC-DQM) Method

The DQM is a numerical technique used for solving differential equations by approximating derivatives as weighted sums of function values at discrete points. Unlike traditional finite difference methods, which rely on local approximations, DQM employs a global approach, using all grid points in the domain to estimate derivatives. This leads to high accuracy with fewer grid points, making it an efficient alternative to finite difference, finite element, and spectral methods.

The origins of DQM trace back to the early 1970s when Richard Bellman and his colleagues introduced it as an extension of the quadrature concept used in numerical integration [6-7]. Bellman's idea was to approximate derivatives similarly to how numerical integration approximates integrals, using a weighted sum of function values [8-9]. Initially, DQM gained attention in computational mathematics but was later widely adopted in engineering disciplines, particularly in solving structural, fluid dynamics, and heat transfer problems. Over the decades, researchers refined the method by deriving weighting coefficients using polynomial-based interpolation, such as Lagrange polynomials, and expanding it to handle more complex boundary conditions and nonlinear problems.

In computational mathematics, a variety of basis functions have been utilized to numerically solve equations such as the Burgers equation, nonlinear Schrödinger equation, multi-dimensional convection–diffusion equations, telegraph equation, sine–Gordon equation, and Fisher's reaction–diffusion equation [10-16]. The exponential modified cubic B-spline differential quadrature method (EMCB-DQM) has been introduced as an effective technique for these purposes. However, its application has been limited, primarily due to the sensitivity of the parameter λ within the EMCB basis functions. Traditionally, λ has been assigned arbitrary values through trial-and-error methods, often leading to unstable and unreliable results.

To address this challenge, recent research has integrated the LOOCV technique with EMCB-DQM to systematically determine the optimal value of ε. LOOCV is a robust method for model evaluation and parameter selection, as it provides an almost unbiased estimate of generalization performance. By employing LOOCV, researchers have achieved a more systematic and reliable approach to parameter optimization, enhancing the stability and accuracy of numerical solutions.

The efficacy of this integrated approach has been demonstrated in studies such as [17] by Rani and Arora. These studies highlight the advantages of combining LOOCV with EMCB-DQM, providing a more systematic approach to parameter optimization. This advancement ensures that the resulting solutions are not only accurate but also reproducible, thereby enhancing the method's reliability and potential applicability across various scientific and engineering problems.

This innovative combination is poised to attract further attention from researchers seeking to improve numerical solution methodologies, offering a more robust framework for solving complex differential equations.

1.2. Method 2: Radial Basis Pseudo-spectral (RB-PS) Method

RB-PS process is an inventive scheme to find the solutions of various PDEs numerically and does not require any meshes. Numerical treatment of various ordinary and partial differential equations requires a successful basis function named RBF. Approaches based on RBFs are extremely novel and fruitful method for various mathematical problems based on geometrical complex domain. Hardy [18] firstly invented the approach based upon multiquadric RBF for interpolation purposes with topological quadric surface. Micchelli [19] made a step forward by demonstrating that multi-quadratic surface interpolation is always solvable. The MQ approach has the benefit of obtaining the interpolant using a linear combination of basis functions that are only dependent on the distance from a specific node, which is known as the center. In order to solve a PDE, Edward Kansa invented the Kansa method [20] in 1990. It first used the multi quadratic, a widely supported interpolant to analyze the behavior of PDEs.

The numerical simulation of a PDE using RBF requires the value of the shape parameter that evaluates the shapes of RBFs and needs to be evaluated precisely. RBFs offer flexibility and efficiency over the complex domain for solving various PDEs. For detailed explanation of RBF with their applications refer the published work Arora et. al [21]. In 2023, RBF-PS method employed to analyze the solutions of PDE with optimization technique [22-23] by optimizing the related parameter of RBF.

Firstly, Fasshauer [24] employed a collocation method using RBF combined with pseudo-spectral approach. Pseudo-spectral methods are known to be perfect solvers in the numerical simulation of PDEs. The pseudo-spectral method with RBF is an advantageous approach for the solution of multi-variate scattered nodes of complicated compounds. The nonlinear partial differential equations were solved by Uddin and Ali [25] using the method RBF-PS. Using the same technique, Uddin [26] proposed the solution to the equation of the same width. The two-dimensional hyperbolic telegraph equations are solved by Rostamy *et al.* [27] with the RBF pseudo-spectral approach. In [28], numerical simulation of Fisher's equation is presented by using the RBF-PS method with two different optimization techniques.

The layout of the article is as follows: Section 2 focus on the LOOCV algorithm with detailed graphical process. Section 3 deals with the numerical solution of two test problems of the Fitzhugh-Nagumo equation with multi-quadric (MQ) RBF and Expo-MCB at different node points using the proposed hybrid approach. Section 4 concludes the outcomes of the proposed method.

2. Leave-one-out-cross-validation (LOOCV) Approach

Leave-one-out-cross-validation (LOOCV) is also one of the best approaches for finding the optimal shape parameter. LOOCV is a technique commonly used to assess the performance and validate the accuracy of predictive models, particularly in the context of machine learning and statistical modelling. Its origins can be traced back to the early days of statistical analysis and cross-validation techniques. It gained prominence in the context of cross-validation as a robust validation technique in the 1970s and 1980s. However, LOOCV can also be adapted and applied to measure the output of the numerical solutions for PDEs. For evaluating the numerical solutions of PDEs, LOOCV with RBF is first used by Rippa [29] for optimizing the shape parameter by minimizing the error function.

Leave-One-Out Cross-Validation (LOOCV) algorithm

The LOOCV method provides a way to evaluate how well a machine learning algorithm will perform when it is used to make predictions. Here's how the LOOCV procedure works:

- The dataset is divided into n subsets, where n is the total number of observations in the dataset.
- In each iteration, one observation is set aside as the test set, and the remaining $n-1$ observations are used to train the model.
- The model is then tested on the single observation that was left out, and this process is repeated n times, ensuring each observation serves as a test set once.
- The performance of the model is evaluated by calculating the average error rate across all iterations.

3. Applications of Numerical Scheme

This section studies the numerical solutions of the FN equation using the RBF-PS method and Expo-MCB-DQ method, which divides the numerical simulation into two parts. In the first part, the approximated derivatives are determined using basis functions, and in the second part, results are derived for the obtained equation with MATLAB using the LOOCV algorithm. Also, this part focuses on the comparative analysis of derived results for i= 1 to N.

$$L_\infty = \max\left(\left|u_{exact}(x_{i,}t) - u(x_{i,}t)\right|\right);$$

$$L_2 = \sqrt{h\sum_{i=1}^{N}\left|u_{exact}(x_{i,}t) - u(x_{i,}t)\right|^2};$$

$$L_{rms} = \sqrt{\frac{1}{\mathrm{N}}\sum_{\mathrm{i}=1}^{\mathrm{N}}\left|u_{exact}(x_{i,}t) - u(x_{i,}t)\right|^2};$$

Example 1: The non-linear general Fitzhugh-Nagumo equation (1) of dimension one in the domain [-10, 10] with exact solution taken from [30]

$$u(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{1}{2\sqrt{2}}\left(x - \frac{2\zeta - 1}{\sqrt{2}}t\right)\right); \qquad x\epsilon[-10, 10]\&\ t \geq 0$$

With boundary conditions taken from the exact solution and the following initial conditions:

$$u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{x}{2\sqrt{2}}\right);$$

A numerical solution of the general FN equation by RBF-PS method is obtained at a = −10, b = 10 $and\ \zeta$=0.75 at various time intervals, and the derived results are compared with the findings of EMCB-DQM. The L_∞, L_2 and RMS error are calculated in Table 1 for time intervals 0.2; 0.5; 1; 1.5; 2; 3 using MQ RBF and EMCB-DQM the optimal value of the parameter $\lambda = 0.889457$ and 0.100050 by RB-PS approach using LOOCV for results at N=101. Figure 1 presents the numerical simulation of solution of FN equation at different time intervals with N= 51 and Δt = 0.01.

Table 1. Comparison of error norms of problem 1 at N= 101 and $\boldsymbol{\Delta t = 0.0001}$

T	$\boldsymbol{L_{RMS}}$	$\boldsymbol{L_\infty}$	$\boldsymbol{L_{RMS}}$	$\boldsymbol{L_\infty}$
	Expo-MCB-DQ Method		**RBF-PS Method**	
0.2	2.6535e-11	4.3470e-05	5.8391e-05	7.3508e-04
0.5	1.8163e-10	1.1288e-04	2.4955e-04	2.2319e-03
1.0	8.2167e-10	2.4076e-04	8.8015e-04	3.0503e-03
1.5	2.0886e-09	3.8563e-04	1.9422e-03	7.8870e-04
2.0	4.2261e-09	5.4974e-04	3.3972e-03	1.33359e-04
3.0	1.2573e-08	9.4619e-04	6.8676e-03	5.9699e-06

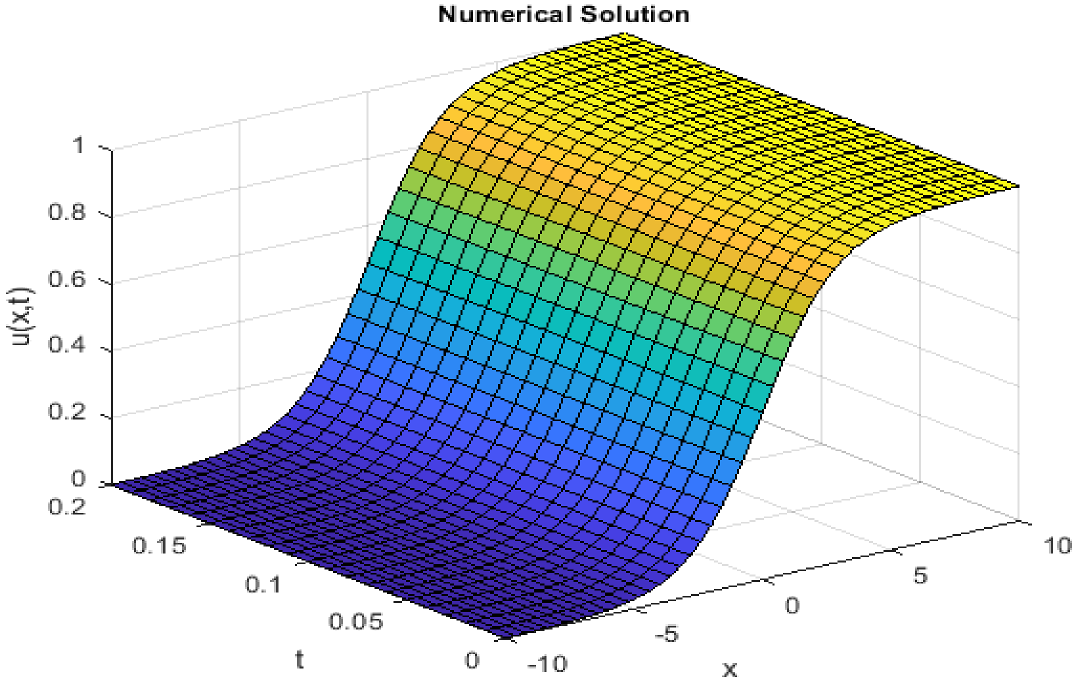

Figure 1. Numerical simulation of problem 1 at N= 51 andΔt = 0.01.

Problem 2: Consider Newell-whitehead equation *i.e.*, special type of Fitzhugh-Nagumo equation (1) with ζ=−1. Its analytic solution [30] considered as

$$u(x,t)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-1}{2\sqrt{2}}\left(x-\frac{3}{\sqrt{2}}t\right)\right);$$

With initial conditions as follows:

$$u(x,0)=\frac{1}{2}+\frac{1}{2}\tanh\left(\frac{-x}{2\sqrt{2}}\right);$$

and boundary conditions taken from the exact solution. For problem 2, Table 2 represents the different error norms calculated by Expo-MCB-DQ method with parameter value $\lambda =$ 0.999934 using LOOCV that are comparable with the results derived by RBF-PS method with cubic matern RBF at different time intervals as 0.001, 0.002 and 0.003 with N=21 with parameter value 0.401232. The graphical presentation of numerical solution is shown by figure 2.

Table 2. $\boldsymbol{L_\infty, L_2}$**and** $\boldsymbol{L_{RMS}}$ errors for Problem 2 with ζ=−1 and N=21.

T	L_∞	L_2	L_{RMS}	L_∞	L_2	L_{RMS}
	Expo-MCB-DQM			**RBF-PS Method**		
0.001	1.2728e-06	2.2883e-12	1.0402e-13	8.7088e-09	2.5076e-08	1.9004e-09

0.002	2.5475e-06	9.1534e-12	4.1606e-13	6.7952e-08	2.3994e-08	5.2359e-09
0.003	3.8241e-06	2.0595e-11	9.3615e-13	4.5460e-08	1.2648e-07	9.9202e-09

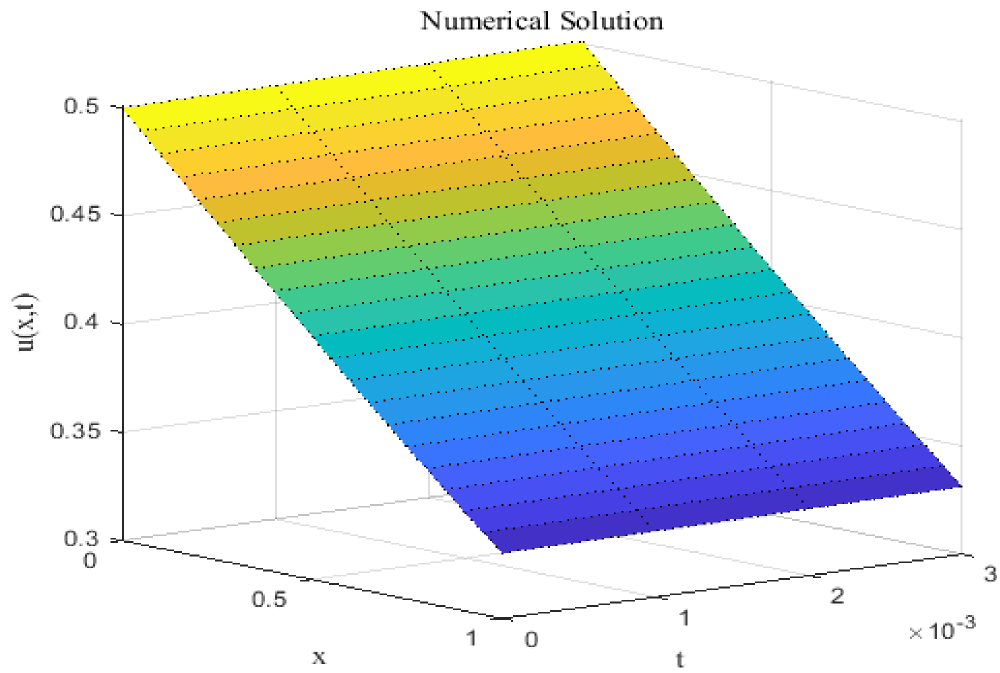

Figure 2. Numerical simulation of problem 2 with N=21.

4. Conclusions

In this paper, two effective numerical techniques are utilized to solve the Fitzhugh-Nagumo equation, an important model in various real-life applications. The first method is the Exponential modified cubic B-Spline differential quadrature method (ECB-DQM), and the second is the Radial basis function Pseudo-spectral method (RB-PS). Both methods incorporate an efficient statistical approach to determine the optimal parameter values for the basis functions, which is a novel contribution of this study.

A comparative analysis of these techniques reveals that the Exponential cubic B-Spline DQM performs slightly better than the Radial Basis Pseudo-Spectral Method. This finding highlights the potential of these numerical approaches for future applications in science and engineering. Additionally, this study emphasizes the significance of the Leave-One-Out Cross-Validation (LOOCV) technique in optimizing parameter selection, further improving the accuracy and efficiency of these methods.

References

1. Nagumo, J., Arimoto, S., & Yoshizawa, S. (1962). An active pulse transmission line simulating nerve axon. *Proceedings of the IRE, 50*(12), 2061-2070. https://doi.org/10.1109/JRPROC.1962.288235
2. FitzHugh, R. (1961). Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal, 1*(6), 445-466. https://doi.org/10.1016/S0006-3495(61)86902-6
3. Zorzano, M. P., & Vazquez, L. (2003). Emergence of synchronous oscillations in neural networks excited by noise. *Physica D: Nonlinear Phenomena, 179*, 105-114. https://doi.org/10.1016/S0167-2789(03)00018-5
4. Argentina, M., Coullet, P., & Krinsky, V. (2000). Head-on collisions of waves in an excitable Fitzhugh-Nagumo system: A transition from wave annihilation to classical wave behavior. *Journal of Theoretical Biology, 1*, 47-52. https://doi.org/10.1006/jtbi.2000.2084
5. Aronson, D. G., & Weinberger, H. F. (1978). Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics, 30*, 33-76. https://doi.org/10.1016/0001-8708(78)90095-7
6. Tamsir, M., Srivastava, V. K., Dhiman, N., & Chauhan, A. (2018). Numerical computation of nonlinear Fisher's reaction–diffusion equation with exponential modified cubic B-spline differential quadrature method. *International Journal of Applied and Computational Mathematics*, *4*(1), 1-13. https://doi.org/10.1007/s40819-017-0437-y
7. Singh, B. K., & Kumar, P. (2018). An algorithm based on a new DQM with modified exponential cubic B-splines for solving hyperbolic telegraph equation in (2 + 1) dimension. *Nonlinear Engineering*, *7*(2), 113-125. https://doi.org/10.1515/nleng-2017-0106
8. Rani, R., Arora, G., Emadifar, H., & Khademi, M. (2023). Numerical simulation of one-dimensional nonlinear Schrödinger equation using PSO with exponential B-spline. *Alexandria Engineering Journal*, *79*, 644-651. https://doi.org/10.1016/j.aej.2023.08.050
9. Arora, G., Rani, R., & Emadifar, H. (2022). Numerical solutions of nonlinear Schrödinger equation with applications in optical fiber communication. *Optik, 266*, 169661. https://doi.org/10.1016/j.ijleo.2022.169661
10. Bellman, R. E., & Casti, J. (1971). Differential quadrature and long-term integration. *Journal of Mathematical Analysis and Applications, 34*, 235-238. https://doi.org/10.1016/0022-247X(71)90019-1
11. Bellman, R. E., Kashef, B. G., & Casti, J. (1972). Differential quadrature: A technique for the rapid solution of nonlinear partial differential equation. *Journal of Computational Physics, 10*, 40-52. https://doi.org/10.1016/0021-9991(72)90033-9
12. Bert, C. W., Jang, S. K., & Striz, A. G. (1988). Two new approximate methods for analyzing free vibration of structural components. *AIAA Journal, 26*, 612-618. https://doi.org/10.2514/3.9937
13. Bert, C. W., & Malik, M. (1996). Differential quadrature in computational mechanics: A review. *Applied Mechanics Reviews, 49*(1), 1-27. https://doi.org/10.1115/1.3101922
14. Tamsir, M., Srivastava, V. K., & Jiwari, R. (2016). An algorithm based on exponential modified cubic B-spline differential quadrature method for nonlinear Burgers' equation. *Applied Mathematics and Computation, 290*, 111-124. https://doi.org/10.1016/j.amc.2016.05.048

15. Shukla, H. S., & Tamsir, M. (2018). An exponential cubic B-spline algorithm for multi-dimensional convection-diffusion equations. *Alexandria Engineering Journal, 57*(3), 1999-2006. https://doi.org/10.1016/j.aej.2017.04.011
16. Msmali, A. H., Tamsir, M., & Ahmadini, A. A. H. (2021). Crank-Nicolson-DQM based on cubic exponential B-splines for the approximation of nonlinear Sine-Gordon equation. *Ain Shams Engineering Journal*, *12*(4), 4091-4097. https://doi.org/10.1016/j.asej.2021.04.004
17. Rani, R., Arora, G., & Bala, K. (2024). Numerical solution of one-dimensional nonlinear Sine–Gordon equation using LOOCV with exponential B-spline. *Computational and Applied Mathematics*, *43*, 188. https://doi.org/10.1007/s40314-024-02672-z
18. Hardy, R. L. (1971). Multiquadric equations of topography and other irregular surfaces. *Journal of Geophysical Research*, *76*(8), 1905-1915. https://doi.org/10.1029/JB076i008p01905
19. Micchelli, C. A. (1984). Interpolation of scattered data: Distance matrices and conditionally positive definite functions. In S. P. Singh, J. W. H. Burry, & B. Watson (Eds.), *Approximation theory and spline functions* (pp. 143-145). NATO ASI Series (Series C: Mathematical and Physical Sciences), Vol 136. Springer.
20. Kansa, E. J. (1990). Multiquadrics—A scattered data approximation scheme with applications to computational fluid-dynamics—II solutions to parabolic, hyperbolic and elliptic partial differential equations. *Computational Mathematics and Applications*, *19*(8), 147-161. https://doi.org/10.1016/0898-1221(90)90271-K
21. Arora, G., Bala, K., Emadifar, H., & Khademi, M. (2023). A review of radial basis function with applications explored. *Journal of the Egyptian Mathematical Society*, *31*(1), 6. https://doi.org/10.1186/s42787-023-00198-0
22. Bala, K., Arora, G., Emadifar, H., & Khademi, M. (2023). Applications of particle swarm optimization for numerical simulation of Fisher's equation using RBF. *Alexandria Engineering Journal*, *84*, 316-322. https://doi.org/10.1016/j.aej.2023.08.050
23. Arora, G., Bala, K., Emadifar, H., & Khademi, M. (2023). A comparative study of particle swarm optimization and artificial bee colony algorithm for numerical analysis of Fisher's equation. *Discrete Dynamics in Nature and Society*, *2023*. https://doi.org/10.1155/2023/9964744
24. Fasshauer, G. E. (2005). RBF collocation methods as pseudospectral methods. *WIT Transactions on Modelling and Simulation, 39*.
25. Uddin, M., & Ali, S. (2012). RBF-PS method and Fourier pseudospectral method for solving stiff nonlinear partial differential equations. *Mathematical Sciences Letters*, *2*(1), 55-61.
26. Uddin, M. (2013). RBF-PS scheme for solving the equal width equation. *Applied Mathematics and Computation*, *222*, 619-631.
27. Rostamy, D., Emamjome, M., & Abbasbandy, S. (2017). A meshless technique based on the pseudospectral radial basis functions method for solving the two-dimensional hyperbolic telegraph equation. *The European Physical Journal Plus, 132*. https://doi.org/10.1140/epjp/i2017-11529-2
28. Bala, K., Arora, G., & Mishra, S. (2024). Particle swarm optimization for solving nonlinear Fisher's equation. *International Journal of Mathematical, Engineering & Management Sciences*, *9*(5).

29. Rippa, S. (1999). An algorithm for selecting a good value for the parameter in radial basis function interpolation. *Advances in Computational Mathematics, 11*, 193-210. https://doi.org/10.1023/A:1018975906542
30. Bhatia, G. S., & Arora, G. (2020). Radial Basis Function Pseudo-spectral Method for Solving Standard Fitzhugh-Nagumo Equation. *International Journal of Mathematical, Engineering and Management Sciences*, *5*, 1488-1497.