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Numerical solutions of Fredholm integral equations via second kind of Chebyshev wavelets

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Abstract

This research paper is devoted to finding the numerical solutions of the Fredholm integral equations with the help of basis functions of the second kind of Chebyshev wavelets. First of all, convert the Fredholm integral equation into differential equations via Leibnitz rule of integration and then apply classical Chebyshev wavelet technique. Some numerical examples have been performed to illustrate the accuracy and simplicity of the proposed scheme.

Keywords: Chebyshev wavelets; Chebyshev wavelets of the second kind; Numerical differentiation; Numerical integration Fredholm integral equations; Test examples.

1. Introduction

Fredholm integral equations are used in many areas like physics, biology, and engineering to describe complex systems. In scattering theory and quantum mechanics, they help find wave functions and understand how particles interact. These equations are important in image and signal processing for removing blur and rebuilding images. In electromagnetics, these equations are used to study how waves travel, which is helpful for designing antennas and radar systems. These equations are also used to analyze stress and strain in structures, to model fluid flow, to describe how particles interact in statistical physics, to study heat transfer in thermodynamics, and to understand how populations change in ecology. In finance, they can be applied to price financial derivatives. They are also useful in acoustics and medical imaging for modeling sound and reconstructing images. As they can be applied in so many different ways, Fredholm integral equations are valuable in both theoretical research and practical work.

Many numerical methods have been developed to solve Fredholm integral equations. Like, solutions have been obtained using wavelet basis functions and collocation methods in [1, 11, 18]. Some researchers have applied the integral mean value theorem to find numerical solutions in [2]. In [3], the authors described a semi-analytical approach for solving integral equations. Other techniques include wavelet-based methods, direct quadrature methods, and boundary integral methods for solving both linear and nonlinear problems [4, 5, 8, 17]. Different

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techniques have been presented for the solution Mixed Fredholm and Volterra integral equations in [6, 9, 14]. A brief study focused on solution of differential equations as well as integral equations has been elaborated in [7, 13, 15, 19]. Authors presented a novel technique based on Haar wavelets for solving integral equations in [10]. Numerical algorithm has been described for the solution of Volterrra integral equations by utilizing Legendre wavelets in [12]. Numerical solution of first kind Fredholm integral equations has been discussed in [16]. Several analytical techniques relevant to present study are discussed in the book in [20]. With the use of Chebyshev wavelets, numerical differentiation and numerical integration has been solved in [21, 22].

2. Second kind of Chebyshev wavelets and its properties

Over recent decades, numerical methods based on wavelets have become widely adopted for solving a range of problems in engineering, science, and technology. Wavelets consist of functions created by shifting and scaling a basic proto-type function, called mother wavelet. By continuously adjusting the scale parameter ℓ and the shift parameter δ, a family of continuous wavelets is generated as follows:

$$\varphi_{\ell,\delta}(t) = |\ell|^{-1/2}\varphi\left(\frac{t-\delta}{\ell}\right), \quad \ell,\delta \in R, \quad \ell \neq 0$$

The second kind Chebyshev wavelets depends upon four parameters k, p, q, t and are denoted as $\varphi_{p,q} = \varphi(k,p,q,t)$. Here, k denotes positive integer and p ranges as $p = 1,2,3,4,\ldots,2^{k-1}$, q represents degree of the Chebyshev polynomials, and the normalized time variable denoted as t. For the interval [0,1), $\varphi_{p,q}$ is defined as follows:

$$\varphi_{p,q}(t) = \begin{cases} 2^{\frac{k}{2}}\tilde{U}(2^{k}t - 2p + 1), & \frac{p-1}{2^{k-1}} \le t \le \frac{p}{2^{k-1}} \\ 0\,, & otherwise \end{cases} \tag{1}$$

where

$$\tilde{U}_q(t) = \sqrt{\frac{2}{\pi}}\,U_q(t), \tag{2}$$

where $\mathcal{M}$ denotes integer, which is fixed and $q = 0,1,2,3,4,\ldots,\mathcal{M}-1$. Eq. (1) describes the orthonormality property. In this context, $U_q(t)$ represents the Chebyshev polynomials having degree q, which are orthogonal w.r.t. the weight function $\omega(t) = \sqrt{1-t^2}$ over the interval $-1 \le t \le 1$. The following recursive formula is satisfied by these polynomials:

$$U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{q+1}(v) = 2tU_q(t) - U_{q-1}(t), \qquad q = 1,2,3,4,\ldots\ldots$$

It should be noted that for second kind Chebyshev wavelets, the weight function is modified by dilation and translation, and can be written as $\omega_p(t) = \omega(2^k t - 2p + 1)$. In this work, we compute the integral of second kind Chebyshev wavelets functions for $k = 1$, $\mathcal{M} = 6$. The six basis functions over the interval $0 \le t \le 1$ are as follows:

$$\varphi_{1,0}(t) = \frac{2}{\sqrt{\pi}},$$

$$\varphi_{1,1}(t) = \frac{2}{\sqrt{\pi}}(4t - 2),$$
$$\varphi_{1,2}(t) = \frac{2}{\sqrt{\pi}}(16t^2 - 16t + 3),$$
$$\varphi_{1,3}(t) = \frac{2}{\sqrt{\pi}}(64t^3 - 96t^2 + 40t - 4),$$
$$\varphi_{1,4}(t) = \frac{2}{\sqrt{\pi}}(256t^4 - 512t^3 + 336t^2 - 80t + 5),$$
$$\varphi_{1,5}(t) = \frac{2}{\sqrt{\pi}}(1024t^5 - 2560t^4 + 2304t^3 - 896t^2 + 140t - 6).$$

3. Proposed methodology

Consider the Fredholm integral equation

$$u(z) = f(z) + \lambda \int_a^b \mathrm{K}(z,t)u(t)dt \tag{3}$$

with ICs $u(z_0) = u_0$.

Differentiating above Eq. (3) two times w.r.t. z by using Leibniz rule of integration, we obtain

$$u'(z) = f'(z) + \lambda \int_a^b \mathrm{K}'(z,t)u(t)dt \tag{4}$$

and

$$u''(z) = f''(z) + \lambda \int_a^b \mathrm{K}''(z,t)u(t)dt \tag{5}$$

Assume that

$$u''(z) = \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{\mathcal{M}-1} c_{p,q}\varphi_{p,q}(z) \tag{6}$$

Integrating Eq. (6) w.r.t z, from z_0 to z, we obtain

$$u'(z) = u'(z_0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{\mathcal{M}-1} c_{p,q} \int_{z_0}^{z} \varphi_{p,q}(z)\, dz \tag{7}$$

Again, integrating Eq. (7) w.r.t z, from z_0 to z, we obtain

$$u(z) = u(z_0) + z.u'(z_0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{\mathcal{M}-1} c_{p,q} \int_{z_0}^{z}\int_{z_0}^{z} \varphi_{p,q}(z)\, dzdz \tag{8}$$

From Eq. (5) to Eq. (8), we obtain a double integrand value and then applying the initial conditions and converting the equation into a discrete form, we obtain a linear system of equations. This system can be solved with standard numerical methods to find the wavelet coefficients. Substituting these coefficients back into Equation (8) gives the solution $u(z)$. In this work, the numerical computations are conducted for $k = 1$ and $\mathcal{M} = 8$. (In the Chebyshev wavelet method, for controlling the resolution and approximation accuracy of the numerical scheme, the parameters $\mathscr{k}$ and $\mathcal{M}$ play distinct but complementary roles. Here, the parameter $\mathscr{k}$ determines the level of wavelet decomposition, and $\mathcal{M}$ represents the number of Chebyshev polynomials employed for each subinterval. By increasing $\mathcal{M}$ improves the approximation accuracy due to the exponential convergence property of Chebyshev polynomials for smooth functions. Through numerical experimentation, it was observed that $\mathcal{M} = 8$ provides stable and sufficiently accurate results. Hence, the choice $\mathscr{k} = 1$, $\mathcal{M} = 8$ is not arbitrary but is motivated by convergence behavior, computational efficiency, and smoothness of the solution. Similar parameter selections have been widely adopted in the literature on Chebyshev wavelets and spectral methods, where moderate values of $\mathscr{k}$ and $\mathcal{M}$ are often sufficient to achieve accurate approximations for smooth problems. The choice of k is made to achieve an optimal balance between efficiency and accuracy, higher values of $\mathscr{k}$ do not lead to a significant improvement in accuracy but substantially increase computational complexity.)

4. Computational analysis

In this Section, the computational and error analysis of the proposed scheme has been performed for three numerical examples.

Example 1: Let us consider the Fredholm integral equation

$$u(z) = (1 - 2\pi)\cos z + \sin z + 4\int_{0}^{\pi} \cos z \cos t \, u(t) dt, \tag{9}$$

with $u(0) = 1$ as the IC. Thus, the given problem has exact solution of the form:

$$u(z) = \sin z + \cos z$$

Use Leibniz rule of integration to differentiate Eq. (9) w.r.t z, we obtain

$$u'(z) = -(1 - 2\pi)\sin z + \cos z - 4\int_{0}^{\pi} \sin z \cos t \, u(t) dt, \tag{10}$$

Again, differentiating Eq. (10) w.r.t z by using Leibniz rule of integration, we obtain

$$u''(z) = -(1 - 2\pi)\cos z - \sin z - 4\int_{0}^{\pi} \cos z \cos t \, u(t) dt, \tag{11}$$

From Eq. (9) and Eq. (11), we obtain

$$u''(z) + u(z) = 0, \tag{12}$$

with $u(0) = 1,\ u(0) = 1$ as the initial conditions

Integrating Eq. (6) w.r.t z, from 0 to z, we obtain

$$u'(z) = u'(0) + \sum_{p=1}^{2^{\mathscr{k}-1}} \sum_{q=0}^{\mathcal{M}-1} c_{p,q} \int_{0}^{z} \varphi_{p,q}(z)\, dz \tag{13}$$

Again, integrating Eq. (13) w.r.t z, from 0 to z, we obtain

$$u(z) = u(0) + z.u'(0) + \sum_{p=1}^{2^{k-1}} \sum_{q=0}^{\mathcal{M}-1} c_{p,q} \int_0^z \int_0^z \varphi_{p,q}(z)\, dzdz \tag{14}$$

From Eq. (12) – Eq. (14), we obtain

$$\sum_{p=1}^{2^{k-1}} \sum_{q=0}^{\mathcal{M}-1} c_{p,q} \left\{ \varphi_{p,q}(z) + \int_0^z \int_0^z \varphi_{p,q}(z)\, dzdz \right\} = -u(0) - z.u'(0) \tag{15}$$

then applying the initial conditions and converting the Eq. (15) into a discrete form, this leads to a system of equations, which is linear. This system can be solved with standard numerical methods to find the wavelet coefficients. Substituting these coefficients back into Equation (8) gives the solution $u(z)$. In this example, the numerical computations are carried out for $k = 1$ and $\mathcal{M} = 8$.

Table 1: Exact versus Chebyshev wavelet solutions for Example 1

s	**Exact Solution**	**Chebyshev wavelets solution**	**Absolute Error**
0.06250	1.060506828	1.060506828	1.5825×10^{-11}
0.18750	1.168876609	1.168876609	5.2461×10^{-11}
0.31250	1.259006462	1.259006462	8.4924×10^{-11}
0.43750	1.329489940	1.329489940	1.1660×10^{-10}
0.56250	1.379227172	1.379227172	1.463×10^{-10}
0.68750	1.407442026	1.407442026	1.7373×10^{-10}
0.81250	1.413694217	1.413694217	1.9894×10^{-10}
0.93750	1.397886183	1.397886183	2.1760×10^{-10}

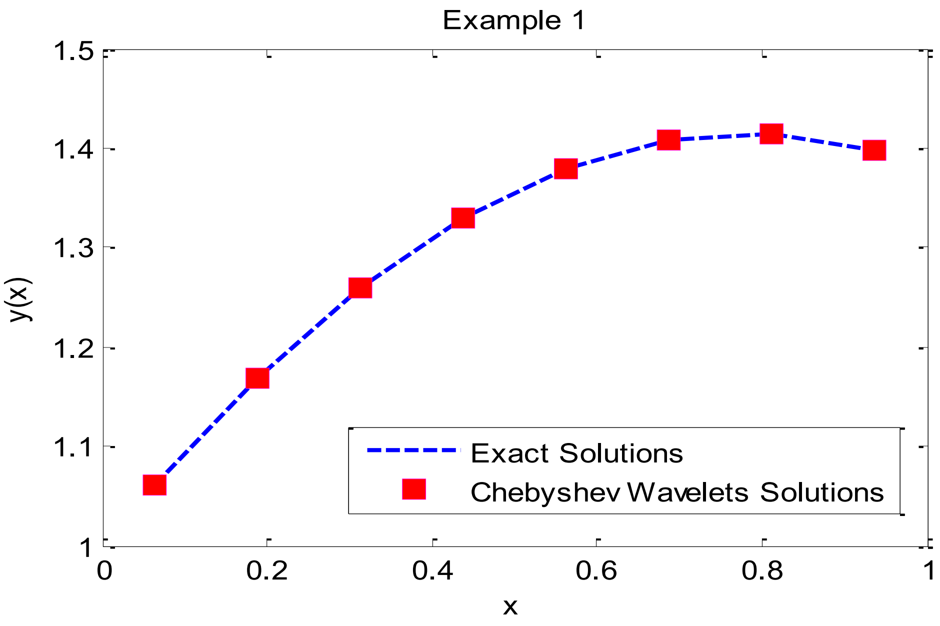

Figure 1: Exact versus Chebyshev wavelet solutions for Example 1

Table 1 compares the exact solutions with those computed using second-kind Chebyshev wavelets. Figure 1 shows how the numerical solutions from the Chebyshev wavelet method match up with the exact solutions for Example 1.

Example 2: Let us consider the Fredholm integral equation

$$u(z) = e^{z} - \frac{4}{3}z + \frac{4}{3}\int_{0}^{1} zt\, u(t)dt, \tag{16}$$

with $u(0) = 1$ as the initial condition. Thus, the given problem has exact solution of the form:

$$u(z) = e^{z}$$

Use Leibniz rule of integration to differentiate Eq. (16) w.r.t z, we obtain

$$u'(z) = e^{z} - \frac{4}{3} + \frac{4}{3}\int_{0}^{1} t\, u(t)dt, \tag{17}$$

Again, differentiating Eq. (17) w.r.t z, by using Leibniz rule of integration, we obtain

$$u''(z) = e^{z}\,, \tag{18}$$

with $u(0) = 1,\ u'(0) = 1$ as the initial conditions.
Now, integrating Eq. (6) w.r.t z, from 0 to z, we obtain

$$u'(z) = u'(0) + \sum_{p=1}^{2^{k}-1}\sum_{q=0}^{\mathcal{M}-1} c_{p,q}\int_{0}^{z} \varphi_{p,q}(z)\, dz \tag{19}$$

Again, integrating Eq. (19) w.r.t z, from 0 to z, we obtain

$$u(z) = u(0) + z.u'(0)\sum_{p=1}^{2^{k}-1}\sum_{q=0}^{\mathcal{M}-1} c_{p,q}\int_{0}^{z}\int_{0}^{z} \varphi_{p,q}(z)\, dzdz$$

This implies

$$u(z) = 1 + z + \sum_{p=1}^{2^{k}-1}\sum_{q=0}^{\mathcal{M}-1} c_{p,q}\int_{0}^{z}\int_{0}^{z} \varphi_{p,q}(z)\, dzdz \tag{20}$$

From Eq. (6) and Eq. (18), we obtain

$$\sum_{p=1}^{2^{k}-1}\sum_{q=0}^{\mathcal{M}-1} c_{p,q}\, \varphi_{p,q}(z) = e^{z} \tag{21}$$

then applying the initial conditions and converting the Eq. (21) into a discrete form, this leads to a system of equations, which is linear. This system can be solved with standard numerical methods to find the wavelet coefficients. Substituting these coefficients back into Equation (20) gives the solution $u(z)$. In this example, the numerical computations are carried out for $k = 1$ and $\mathcal{M} = 8$.

Table 2*:* Exact versus Chebyshev wavelet solutions for Example 2

s	**Exact Solution**	**Chebyshev wavelets solution**	**Absolute Error**
0.06250	1.064494458	1.064494458	1.8715×10^{-11}
0.18750	1.206230249	1.206230249	6.2314×10^{-11}
0.31250	1.366837941	1.366837941	1.0192×10^{-10}
0.43750	1.548830298	1.548830298	1.4217×10^{-10}
0.56250	1.755054656	1.755054656	1.8231×10^{-10}
0.68750	1.988737469	1.988737469	2.2232×10^{-10}
0.81250	2.253534787	2.253534786	2.6302×10^{-10}
0.93750	2.553589458	2.553589457	2.9934×10^{-10}

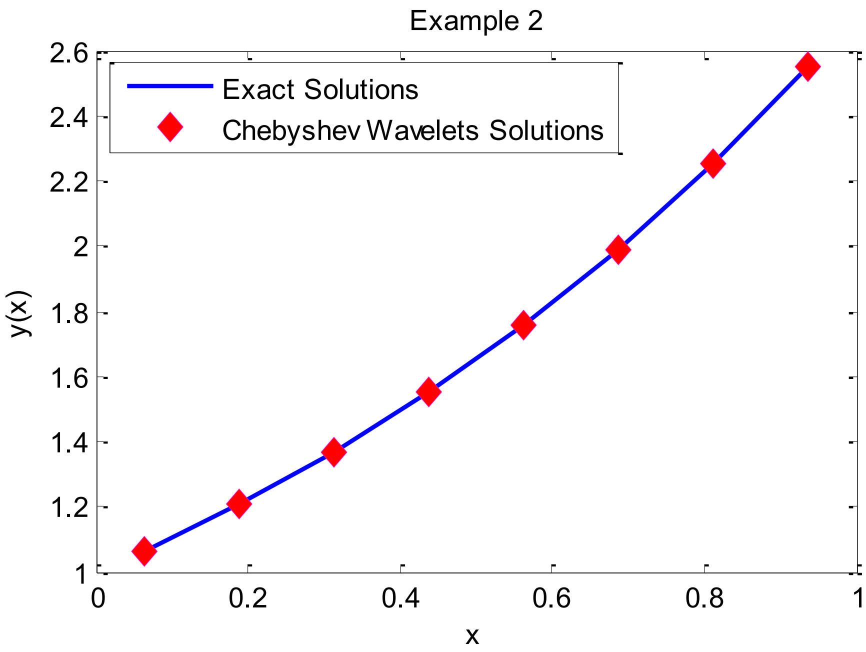

Figure 2: Exact versus Chebyshev wavelet solutions for Example 2

Table 2 compares the exact solutions with those computed using second-kind Chebyshev wavelets. Figure 2 shows how the numerical solutions from the Chebyshev wavelet method match up with the exact solutions for Example 2.

Example 3: Let us consider the following integro-differential equation

$$u'(z) = z + \frac{5}{8} - \int_{0}^{1} t^2 u(t)dt,$$

with $u(0) = 1,\ u'(0) = 0$ as the ICs. Thus, the problem has exact solution of the form:

$$u(z) = \frac{z^2}{2} + 1$$

Table 3: Exact versus Chebyshev wavelet solutions for Example 3

s	**Exact Solutions**	**Chebyshev wavelets solutions**	**Absolute Errors**
0.06250	1.001953125	1.001953125	0.0
0.18750	1.017578125	1.017578125	0.0
0.31250	1.048828125	1.048828125	0.0
0.43750	1.095703125	1.095703125	0.0
0.56250	1.158203125	1.158203125	0.0
0.68750	1.236328125	1.236328125	2.2204×10^{-16}
0.81250	1.330078125	1.330078125	4.4409×10^{-16}
0.93750	1.439453125	1.439453125	4.4409×10^{-16}

Table 3 compares the exact solutions with those computed using second-kind Chebyshev wavelets. Figure 3 shows how the numerical solutions from the Chebyshev wavelet method match up with the exact solutions for Example 3.

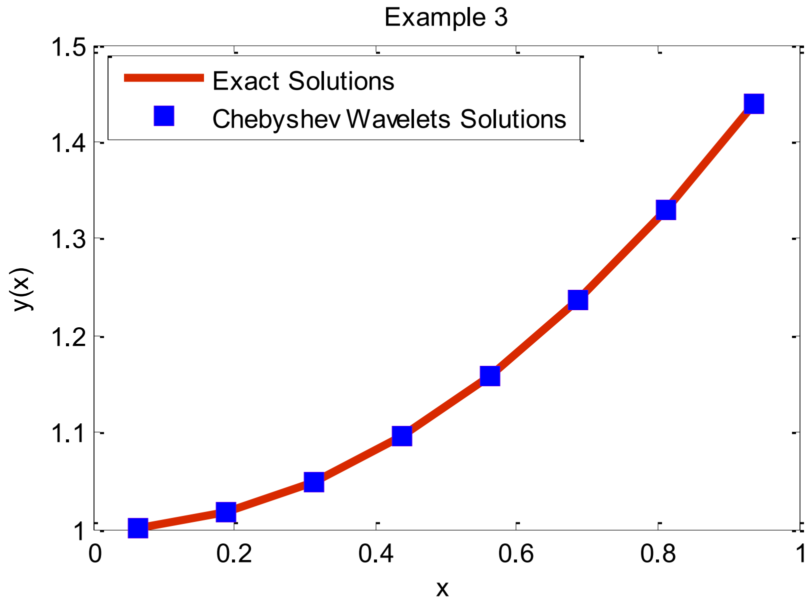

Figure 3: Exact versus Chebyshev wavelet solutions for Example 3.

5. Conclusion

In this work, a numerical scheme is developed for the solution of certain Fredholm integral equations by employing Chebyshev wavelets of the second kind in conjunction with the Leibniz integration rule. By utilizing an integrated formulation, the proposed approach improves the smoothness of the solution and facilitates an efficient wavelet approximation. Numerical experiments demonstrate that Chebyshev wavelets of the second kind yield solutions with high accuracy. Furthermore, as the number of collocation points increases, the numerical results progressively converge to the exact solutions This proposed methodology can be extended to other classes of integral equations and to non-linear equations and multidimensional problems.

Conflict of interest

There is no conflict of interest in this study.

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