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# Conditional Expectations on $L^p$-Spaces Associated with Partial Generalized von Neumann Algebra

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**Abstract**

Weak and unbounded Conditional Expectations have been extended on $L^p$-spaces over Partial Generalized von Neumann Algebras and their properties studied. It was discovered that the Unbounded Conditional Expectations is a proper subspace of the Weak Conditional Expectations. The existence of Conditional Expectations as given by Takesaki [2] was also checked for Partial Generalized von Neumann Algebras on $L^p$-spaces under the modular automorphism group given by $\Delta_{\Omega_0}''^{it}(L^p(N)'_w)'\,\Delta_{\Omega_0}''^{-it} = (L^p(N)'_w)'$, $\forall t \in R$ where $\Delta''_{\Omega_0}$ is the modular operator for the full Hilbert Algebra $(L^p(M)'_w)$, $L^p(M)$ is the Partial Generalized von Neumann Algebra and $L^p(N)$ is the Partial Generalized von Neumann sub-algebra of $L^p(M)$ associated with $L^p$-spaces. Hence the work of Tijjani and Dangana [11] has been extended.

**Keywords:** $L^p$-spaces, Conditional Expectations, von Neumann Algebra, multipliers, Partial *-algebras, partial O*-algebras.

## 1. Introduction

Conditional Expectations is a very important tool in the entire Mathematical field and other Engineering fields especially in Quantum Mechanics where electrons are treated as elements of the given system on which it acts upon. It also has a wide range of application in other applied sciences.

Given a non- empty set $M$ having a sub-set $N$ which satisfies properties of an algebra, Conditional Expectations takes elements from $M$ into $N$; This implies that the domain of Conditional Expectations is not equal to the algebra in which it is acting upon.



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The study of conditional expectations for O*- algebras was first done by Gudder and Hudson [1]. Suppose $N \subseteq M$ is an O*-algebra acting on a dense subspace $D$ of a complex separable Hilbert space $H$ with cyclic and separating vector $\Omega_0$; they defined Conditional Expectations as a map $A \to P_N A\Omega_0$ of $M$ into the closed subspace $H_N$ of $H$, where $P_N$ is the orthogonal projection of $H$ onto $H_N$ . This is the vector Conditional Expectations given by $(N, \Omega_0)$.

Several authors have studied Conditional Expectations on different algebras such as Takesaki, Takakura, Ogi, Inoue, [2, 3, 4, 5]. It is important to check the existence of a Conditional expectation. In fact, Takesaki [2] has shown that Conditional Expectations does not necessarily exist for a general von Neumann algebra. But for semi finite von Neumann algebras, here Conditional Expectations exist if and only if $\Delta_{\Omega 0}^{it} N \Delta_{\Omega 0}^{-it} = N$ where $\Delta$ is the modular automorphism group.

While for $L^p$-spaces on von Neumann Algebras, we refer the reader to the work of Terp [9]. In their unpublished manuscript, Tijjani and Dangana [10 ] took up the study of $L^p$-spaces to partial O*-algebras thereby extending the work of Takakura [5]. Moving in this direction, we extended the work of Tijjani and Dangana [11] on $L^p$-spaces to Partial Generalized von Neumann Algebras.

## 2. Preliminaries

In order to make the paper self-contained, we reproduce the definitions of partial *-algebras, partial O*-algebras and Partial Generalized von Neumann Algebra. For more details on the subject, we refer the reader to [6].

***-algebra:** A ***-**algebra is an algebra $\mathfrak{A}$, together with an involution which enjoys the following properties;

(i) $(x+y)^* = y^* + x^*$

(ii) $(x.y)^* = y^*.x^*$

(iii) $x^{**} = x$

(iv) $(\alpha x)^* = \bar{\alpha} x^*$, for all $x, y \in \mathfrak{A}$, $\alpha \in C$.

**Partial *-algebra:** A partial *-algebra is a complex vector space $\mathfrak{A}$ with an involution $x \to x^*$ (that is a bijection $x^{**} = x$) and a subset $\Gamma \subset \mathfrak{A} \times \mathfrak{A}$ (a binary relation) such that

(i) $(x, y) \in \Gamma$ implies $(y^*, x^*) \in \Gamma$

(ii) $(x, y)$, $(x, z) \in \Gamma$ implies $(x, \alpha y + \beta z) \in \Gamma$, for all $\alpha, \beta \in \boldsymbol{C}$;

(iii) Whenever $(x, y) \in \Gamma$, there exists a product $x.y \in \mathfrak{A}$ with the usual properties of the multiplication: $x.(y + \alpha z) = x.y + \alpha(x.z)$ and $(x.y)^* = y^*.x^*$ for $(x, y), (x, z) \in \Gamma$ and $\alpha \in \boldsymbol{C}$.

The element $e$ of $\mathfrak{A}$ is called a unit if $e^* = e, (e, x) \in \Gamma$ for all $x \in \mathfrak{A}$ and $e.x = x.e = x$, for all $x \in \mathfrak{A}$. Notice that the partial multiplication is not required to be associative. Whenever $(x, y) \in \Gamma$, $x$ is called the left multiplier of $y$ and $y$ is called the right multiplier of $x$ and we write $x \in L(y)$ and $y \in R(x)$. For a subset $\mathfrak{N} \subset \mathfrak{A}$, we write

$$L(\mathfrak{N}) = \bigcap_{x\in\mathfrak{N}} \mathrm{L}(x) \text{ , } R(\mathfrak{N}) = \bigcap_{x\in\mathfrak{N}} R(x).$$

Note that if $\mathfrak{A}$ has no unit, it may always be embedded into a larger partial *-algebra with unit in the standard fashion.

**Partial $\boldsymbol{O}$*-Algebra:** A partial $O$*-algebra is a *-subalgebra $M$ of $L_w^{\dagger}(D, H)$, with identity satisfying the following properties:

$(i)$ $X_1 + X_2$, $X_1, X_2 \in M$, (ii) $\alpha X$, $\alpha \in \boldsymbol{C}, X \in M$. (iii) $X \to X^{\dagger} = X^* \upharpoonright D$, (iv) $X_1 \square X_2 = X_1^{\dagger *} X_2$, defined whenever $X_1 \in L^w(X_2)$ or $X_2 \in R^w(X_1)$, that is if and only if $X_2 D \subset D(X_1^{\dagger *})$ and $X_1^* D \subset D(X_2^*)$, for all $X^{\dagger} \in M$, $X_1, X_2 \in M$

***-Representation:** A *- representation of a partial *- algebra $\mathfrak{A}$ is a *- homomorphism of $\mathfrak{A}$ into $L^{\dagger}(D, H)$, satisfying $\pi(e) = I$, whenever $e \in \mathfrak{A}$, that is,

(i) $\pi$ is linear;

(ii) $x \in L^w(y)$ in $\mathfrak{A}$ implies $\pi(x) \in L^w(\pi(y))$ and $\pi(x) \square \pi(y) = \pi(xy)$;

(iii) $\pi(x^*) = \pi(x)^{\dagger}$ for every $x \in \mathfrak{A}$

**3. Properties of Conditional Expectations on von Neumann Algebra**

Here we state the properties of Conditional Expectations on von Neumann Algebra. For the properties of the classical Conditional Expectations, [8] has done it extensively.

Let $N \subseteq M$ be a von Neumann Algebra on a separable Hilbert space $H$ with a faithful normal state ω and a cyclic vector $\Omega_0$ in $H$; then a map $E$ of $M$ onto $N$ is said to be a Conditional Expectations of $M$ onto $N$ if it satisfies the following properties:

1. $E$ is linear,
2. $E(A)^* = E(A^*)$ for all $A \in M$
3. $E(X) = X$, for all $X \in N$,
4. $E(A^* A) \geq 0$, for all $A \in M$
5. $E(A^* A) \leq E(A^*) E(A)$, for all $A \in M$.
6. $E(XAY) = XE(A)Y$, for all $A \in M$, $X, Y \in N$
7. $E(E(A)X) = E((A)E(X)) = E(A)E(X)$, for all $A \in M, X \in N$ or $X \in M, A \in N$.
8. $\omega_{\Omega_0}\big(E(A)\big) = \omega_{\Omega_0}(A)$, for all $A \in M$.

Remark: [7] has proved that every projection of norm one of a $C^*$-algebra onto its $C^*$-subalgebra enjoys properties 4-6.

## 4. Existence of Conditional Expectations in von Neumann Algebra

Let $N \subseteq M$ be a von Neumann Algebra acting on a separable Hilbert space $H$ with a faithful normal semifinite weight $\omega$ on $M_+$. Then the following two statements are equivalent.

i. $N$ is invariant under the modular automorphism group $\sigma_t$ associated with $\omega$,
ii. There exists a $\sigma$ -weakly continuous faithful projection $E$ of norm one from $M$ onto $N$ such that
$\omega(X) = \omega \circ E(X)$, for every $M_\omega$.

**Definition 4.1** Let $M_0$ be a von Neuman Algebra on $H$ such that $M_0'D \subset D$. A partial $O^\star$-algebra $M$ on $D$ is called a Partial Generalized von Neuman Algebra on $D$ over $M_0'$ if $D = \cap_{X \in M} D(\overline{X})$, and $M = [M_0 \upharpoonright D]^{s*}$. Supposed that $M$ is a Partial Generalized von Neuman Algebra on $D$ over $M_0'$. Then it follows that;

$$M''_{w\sigma} = \{X \in L^\dagger(D,H) : < CX\xi, \eta > = < C\xi, X^\dagger \eta >, \text{for each } C \in M_c', \xi, \eta \in M\} \equiv \{X \in L^\dagger(D,H) : \bar{X}\eta M_0\}.$$

For construction of Partial Generalized von Neumann Algebra, see the work of Tijjani and Dangana [11]

## 5. Conditional Expectations on $L^p$-Spaces over Partial Generalized von Neumann Algebra

In this section, let $L^p(M)$ be a Partial Generalized von Neumann Algebra associated with $L^p$-spaces on $D$ over $L^p(M_0)$ (where $L^p(M_0)$ is a von Neumann Algebra associated with $L^p$-spaces on $H$) with a strongly cyclic and separating vector $\Omega_0 \in D$ and let $L^p(N)$ be a Partial Generalized von Neumann sub-algebra of $L^p(M)$ associated with $L^p$-spaces over $L^p(N_0)$ (where $L^p(N_0)$ is a von Neumann subalgebra over $L^p(M_0)$ associated with $L^p$-spaces).

### 5.1 Weak Conditional Expectations on $L^p$-Spaces .

Let $L^p(N) \subseteq L^p(M)$ be a Partial Generalized von Neumann Algebra associated with $L^p$-spaces on $D$ with a strongly cyclic and separating vector $\Omega_0 \in D$ such that $(L^p(N) \cap R^w(L^p(M)))\, \Omega_0$ is dense in $H_{L^p(N)} \equiv \overline{L^p(N)\Omega_0}$. Then the following lemma is immediate:

**Lemma 5.1** Put

$$\pi_N : (L^p(N) \cap R^w(L^p(M)))\, \Omega_0 \longrightarrow L^\dagger(D,H)$$

$$XY\Omega_0 \longmapsto (X \square Y)\, \Omega_0$$

$D(\pi_{L^p(N)}) = (L^p(N) \cap R^w(L^p(M)))\, \Omega_0,$

$\pi_{L^p(N)}(X)Y\Omega_0 = (X\Box Y)\,\Omega_0,\ \forall X \in L^p(N), \forall Y \in\ L^p(N) \cap R^w(L^p(M)).$

Then $\pi_{L^p(N)}$ is a *-representation of $L^p(N)$ in the Hilbert space $H_{L^p(N)} \equiv \overline{D(\pi_{L^p(N)})}$. We denote by $P_{L^p(N)}$ the projection of $L^2(L^p(M)) \equiv\ H_{L^p(M)}$ onto $L^2(N) \equiv H_{L^p(N)}$.

This projection plays a vital roll in this Research.

**Lemma 5.2** If $P_{L^p(N)}$ and $\pi_{L^p(N)}$ are defined as

$$P_{L^p(N)}: L^2(L^p(M)) \longrightarrow L^2(L^p(N)) \xrightarrow{\pi_{L^p(N)}} L^2(L^p(N))$$

$$X\Omega_0 \longmapsto P_{L^p(N)}X\Omega_0 = XP_{L^p(N)}\Omega_0.$$

Then it holds that

$$P_{L^p(N)}D \subset D^*(\pi_{L^p(N)})$$

And

$$\pi^*_{L^p(N)}(X)\ P_{L^p(N)}\Omega_0 = P_{L^p(N)}X\Omega_0, \forall\ X \in L^p(N), \Omega_0 \in D.$$

**Proof.**

$$\left\langle (X\Box Y)\ \Omega_0 \middle| P_{L^p(N)}\Omega \right\rangle = \left\langle\ X^\dagger Y\Omega_0 \middle| \Omega \right\rangle = \langle\ Y\Omega_0 | X\Omega \rangle = \left\langle\ Y\Omega_0 \middle| P_{L^p(N)}X\Omega \right\rangle$$

And so

$$P_{L^p(N)}D \subset D^*(\pi_{L^p(N)})$$

And

$$\pi^*_{L^p(N)}(X)\ P_{L^p(N)}\Omega_0 = P_{L^p(N)}X\Omega_0, \forall\ X \in L^p(N), \Omega_0 \in D.$$

**Definition 5.1** A map $E_{L^p(N)}$ of $L^p(M)$ into $L^\dagger(D(\pi_{L^p(N)}), H_{L^p(N)})$ is said to be a weak Conditional Expectations of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$ if it satisfies

$\left\langle\ E_{L^p(N)}(AX\Omega_0) \middle| Y\Omega_0 \right\rangle = \left\langle P_{L^p(N)}(AX)\Omega_0 \middle| Y\Omega_0 \right\rangle, \forall A \in L^p(M), \forall X, Y \in\ L^p(N) \cap R^w(L^p(M)).$

For weak Conditional Expectations, we have the following theorem;

**Theorem 5.2** There exists a unique weak Conditional Expectation $E_{L^p(N)}$ of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$ and $E_{L^p(N)}(A) = P_{L^p(N)}A \restriction\ D(\pi_{NL^p(N)}), \forall A \in L^p(M).$

The weak Conditional Expectation $E_{L^p(N)}$ of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$ has the following properties

1. $E_{L^p(N)}$ is linear,
2. $E_{L^p(N)}$ is a projection, that is $E_{L^p(N)}(A)^\dagger = E_{L^p(N)}(A^\dagger), \forall A \in L^p(M)$,
3. $E_{L^p(N)}(X) = X, \forall X \in L^p(N)$,
4. $E_{L^p(N)}(A^\dagger \square A) \geq 0, \forall A \in L^p(M)$ such that $A^\dagger \square A$ is well-defined,
5. $E_{L^p(N)}(A^\dagger \square A) = E_{L^p(N)}(A^\dagger) \square E_{L^p(N)}(A), \quad \forall A \in L^p(M)$ such that $A^\dagger \square A$ and $E_{L^p(N)}(A^\dagger) \square E_{L^p(N)}(A)$, are well-defined,
6. $E_{L^p(N)}(A \square X) = E_{L^p(N)}(A) \square X$, for any $A \in L^p(M), X \in L^p(N) \cap R^w(L^p(M))$ and $E_{L^p(N)}(A) \square X$ is well-defined,
7. $E_{L^p(N)}(X \square A) = X \square E_{L^p(N)}(A)$, for any $A \in L^p(M) \cap R^w(L^p(N)), X \in L^p(N)$,
8. $\omega_{\Omega_0}(E_{L^p(N)}(A)) = \omega_{\Omega_0}(A)$, for all $A \in L^p(M)$.

**Proof** . We know that $E_{L^p(N)}(A)$ is a linear map of $D(\pi_{L^p(N)})$ into $D^\star(\pi_{L^p(N)})$ for any $A \in L^p(M)$, and furthermore, we have $E_{L^p(N)}(A)^\dagger = E_{L^p(N)}(A^\dagger)$, for all $A \in L^p(M)$. So $E_{L^p(N)}$ is a map of $L^p(M)$ into $L^\dagger(D(\pi_{L^p(N)}), H_{L^p(N)})$.

Since

$$\langle E_{L^p(N)}(AX\Omega_0) | Y\Omega_0 \rangle = \langle P_{L^p(N)}(AX)\Omega_0 | Y\Omega_0 \rangle, \forall A \in L^p(M), \forall X, Y \in L^p(N) \cap R^w(L^p(M)).$$

$E_{L^p(N)}$ is a weak Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$, $E(A) = E_{L^p(N)}(A)$, for each $A \in L^p(M)$. Thus we have shown the existence and uniqueness of weak Conditional Expectation. The conditions 3-5 follow, since $E_{L^p(N)}(A) = P_{L^p(N)} A \upharpoonright D(\pi_{L^p(N)}), \forall A \in L^p(M)$. This completes the proof.

**6. Unbounded Conditional Expectations on $L^p$-Spaces.**

Let $L^p(N) \subseteq L^p(M)$ be a Partial Generalized von Neumann Algebra associated with $L^p$-spaces on $D$ with a strongly cyclic and separating vector $\Omega_0 \in D$ such that $(L^p(N) \cap R^w(L^p(M)))\, \Omega_0$ is dense in $H_{L^p(N)}$.

**Definition 6.1** A map $E: D(E) \subseteq L^p(M)$ onto $L^p(N)$ is said to be an Unbounded Conditional Expectations of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$ if

i. The domain $D(E)$ of $E$ is a †-invariant subspace of $L^p(M)$ containg $L^p(N)$,
ii. $E$ is a projection, that is hermitian $E(A)^\dagger = E(A^\dagger)$, for $A \in D(E)$ and $E(X) = X, \forall X \in L^p(N)$,
iii. $E(A \square X) = E(A) \square X$, for any $A \in D(E), X \in L^p(N) \cap R^w(L^p(M))$

iv. $E(X\square A) = X\square E(A)$, for any $A \in D(E) \cap R^w(L^p(N)), X \in L^p(N)$
v. $\omega_{\Omega_0} E(A)) = \omega_{\Omega_0}(A)$, for all $A \in D(E)$,

Remark 6.1 If $D(E) = L^p(M)$ then $E$ is said to be a weak Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$.

Note that the Unbounded Conditional Expectation $E$ is a subspace of the weak Conditional Expectation $E_N$ of $(L^p(M), \Omega_0)$ with respect to $N$. That is if $E_{L^p(N)}: L^p(M) \longrightarrow L^p(N)$, then $E: D(E) \subset L^p(M) \longrightarrow L^p(N)$, also $E_{L^p(N)} \upharpoonright D(E) = E$.

For Unbounded Conditional Expectation, we have the following lemma

**Lemma 6.1** Let $E$ be an Unbounded Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$. Then

$$E(AX)\Omega_0 = P_{L^p(N)} AX\Omega_0, \forall\, A \in D(E), X \in\ L^p(N) \cap R^w(L^p(M)).$$

**Proof.**

$$\begin{aligned}\langle E\,(AX)\Omega_0 | Y\Omega_0\rangle &= \langle\, E(A\square X)\Omega_0 | Y\Omega_0\rangle = \left\langle\, E(Y^\dagger \square A\square X)\Omega_0 \middle| \Omega_0\right\rangle = \left\langle\, (Y^\dagger \square A\square X)\Omega_0 \middle| \Omega_0\right\rangle \\ &= \langle\, (A\square X)\Omega_0 | Y\Omega_0\rangle = \langle\, (AX)\Omega_0 | Y\Omega_0\rangle = \left\langle\, (AX)\Omega_0 \middle| P_{L^p(N)} Y\Omega_0\right\rangle \\ &= \left\langle\, P_{L^p(N)} AX\Omega_0 \middle| Y\Omega_0\right\rangle\end{aligned}$$

Hence,

$E\,(AX)\Omega_0 = P_{L^p(N)} AX\Omega_0, \forall A \in D(E), X \in\ L^p(N) \cap R^w(L^p(M)).$ □

Let $\mathfrak{J}$ be the set of all Unbounded Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$. Then $\mathfrak{J}$ is an ordered set with the following order $\subset$:

$E_1 \subset E_2$ if and only if $D(E_1) \subset D(E_2),\ E_1(A) = E_2(A), \forall A \in D(\,E_1)$.

**Theorem 6.1** There exists a maximal Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$, and it is denoted by $\mathcal{E}_n$.

**Proof.**

We put

$D(\mathcal{E}_0) \equiv \{A \in L^p(M): A \upharpoonright_{(L^p(N)\cap R^w(L^p(M)))\Omega_0} \in L^p(N) \upharpoonright_{(L^p(N)\cap R^w(L^p(M)))\Omega_0}\}$.

Then for any $A \in D(\mathcal{E}_0)$, there exists a unique map $\mathcal{E}_0$ such that

$$\mathcal{E}_0\,(AX)\Omega_0 = P_{L^p(N)} AX\Omega_0 =\ E(AX)\Omega_0, \forall X \in\ L^p(N) \cap R^w(L^p(M)).$$

It is easily shown that $\mathcal{E}_0$ is an Unbounded Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to$L^p(N)$. Moreover, $\mathcal{E}_0$ is maximal in $\mathfrak{J}$. Indeed, let $E \in \mathfrak{J}$. Take an arbitrary $A \in D(E)$. Then by lemma **6.1** we see that

$$E\,(AX)\Omega_0 = P_{L^p(N)} AX\Omega_0 = E_{L^p(N)}(AX)\Omega_0, \forall X \in L^p(N) \cap R^w(L^p(M)).$$

Which implies that $E(AX) \upharpoonright_{(L^p(N) \cap R^w(L^p(M)))\Omega_0}$. Hence $E \subset \mathcal{E}_0$ and $\mathcal{E}_0$ is maximal in $\mathfrak{J}$. □

Thus, we remark for the weak and for the Unbounded Conditional Expectations $E_{L^p(N)}$ and $E$ that $E_{L^p(N)} = L^p(N)$, $E(D(E)) \neq L^p(N)$ and $E(D(E)) \subset L^p(N)$.

## 7. Existence of Conditional Expectations on $L^p$-Spaces over Partial Generalized von Neumann Algebra

For the existence of Conditional Expectations Partial Generalized von Neumann Algebra, Tijjani and Dangana [10] has obtained the following:

**Theorem 7.1** Let $M$ be a Partial Generalized von Neumann Algebra on $D$ in $H$ with a strongly cyclic and separating vector $\Omega_0 \in D$, and let $N$ be a Partial Generalized von Neumann subalgebra of $M$ satisfying $N'_w\widehat{D}(N) \subset \widehat{D}(N)$, $(N \cap R^w(M))\,\Omega_0$ is essentially self-adjoint for $N$ and $E_N(A) = P_N A \upharpoonright P_N\, D, \forall A \in M''_{wc}$. Then

1. $E_N$ is linear,
2. $E_N$ is hermitian, that is $E_N(A)^\dagger = E_N(A^\dagger), \forall A \in M$,
3. $E_N(X) = X, \forall\, X \in N$,
4. $E_N(A^\dagger \square A) \geq 0, \forall A \in M$ such that $A^\dagger \square A$ is well-defined,
5. $E_N(A^\dagger \square A) = E_N(A^\dagger) \square E_N(A)$, $\forall A \in M$ such that $A^\dagger \square A$ and $E_N(A^\dagger) \square E_N(A)$, are well-defined,
6. $E_N(A \square X) = E_N(A) \square X$, for any $A \in M, X \in N \cap R^w(M)$ and $E_N(A) \square X$ is well-defined,
7. $E_N(X \square A) = X \square E_N(A)$, for any $A \in M \cap R^w(N), X \in N$,
8. $\omega_{\Omega_0}(E_N(A)) = \omega_{\Omega_0}(A)$, for all $A \in M$.
9. $\Delta''^{it}_{\Omega_0}(N'_w)'\,\Delta''^{-it}_{\Omega_0} = (N'_w)', \forall t \in R$ where $\Delta''_{\Omega_0}$ is the modular operator for the full Hilbert Algebra $(M'_w)'$.

Then our extension is as follows:

**Theorem 7.2** Let $L^p(N) \subseteq L^p(M)$ be a Partial Generalized von Neumann Algebra associated with $L^p$-spaces on $D$ with a strongly cyclic and separating vector $\Omega_0 \in D$, satisfying $L^p(N)'_w\widehat{D}(L^p(N)) \subset \widehat{D}(L^p(N))$, $(L^p(N) \cap R^w(L^p(m)))\,\Omega_0$ is essentially self-adjoint for $L^p(N)$ and $E_{L^p(N)}(A) = P_{L^p(N)} A \upharpoonright P_{L^p(N)}\, D, \forall A \in L^p(M)''_{wc}$. Then

1. $E_{L^p(N)}$ is linear,

2. $E_{L^p(N)}$ is hermitian, that is $E_{L^p(N)}(A)^\dagger = E_{L^p(N)}(A^\dagger), \forall A \in L^p(M)$,
3. $E_{L^p(N)}(X) = X, \forall X \in L^p(N)$,
4. $E_{L^p(N)}(A^\dagger \square A) \geq 0, \forall A \in L^p(M)$ such that $A^\dagger \square A$ is well-defined,
5. $E_{L^p(N)}(A^\dagger \square A) = E_{L^p(N)}(A^\dagger) \square E_{L^p(N)}(A), \quad \forall A \in L^p(M)$ such that $A^\dagger \square A$ and $E_{L^p(N)}(A^\dagger) \square E_{L^p(N)}(A)$, are well-defined,
6. $E_{L^p(N)}(A \square X) = E_{L^p(N)}(A) \square X,$ for any $A \in L^p(N), X \in L^p(N) \cap R^w(L^p(M))$ and $E_{L^p(N)}(A) \square X$ is well-defined,
7. $E_{L^p(N)}(X \square A) = X \square E_{L^p(N)}(A)$, for any $A \in L^p(M) \cap R^w(L^p(N)), X \in L^p(N)$,
8. $\omega_{\Omega_0}(E_{L^p(N)}(A)) = \omega_{\Omega_0}(A)$, for all $A \in L^p(M)$.
9. $\Delta_{\Omega_0}''^{it}(L^p(N))' \Delta_{\Omega_0}''^{-it} = (L^p(N)'_w)', \forall t \in R$ where $\Delta''_{\Omega_0}$ is the modular operator for the full Hilbert Algebra $(L^p(M)'_w)'$.

**Proof.**

Let

$$D(E_{L^p(N)}) = \{A \in M : P_{L^p(N)} A \Omega_0 \in L^p(N) \Omega_0\}$$

Then we see that

$P_{L^p(N)} A \Omega_0 = E_{L^p(N)}(A)\, \Omega_0 \in L^p(N) \Omega_0$, for each $A \in L^p(M)$. Hence $D(E_{L^p(N)}) \subset L^p(M)$.

Since $\Omega_0$ is strongly cyclic and separating vector for $L^p(M)$. It follows that for any $A \in D(E_N)$. There exists a unique element $E_{L^p(N)}(A)$ of $L^p(N)$ such that $P_{L^p(N)} A \Omega_0 = E_{L^p(N)}(A)\, \Omega_0$.

Take arbitrary $X \in L^p(N)$, then $\bar{X}$ is affiliated with the von Neumann Algebra $(L^p(N)'_w)'$. And so

$L^p(N)'_w = L^p(N)'_{qw}$.

By the self-adjointness of $L^p(M)$ and $(L^p(N) \cap R^w(L^p(M)))\, \Omega_0$ being dense in $H_{L^p(N)}$, it follows that

$$L^p(N)(L^p(N) \cap R^w(L^p(N)))\, \Omega_0 \subset \overline{(L^p(N) \cap R^w(L^p(M)))\, \Omega_0} = \overline{L^p(N) \Omega_0},$$

where $(L^p(N) \cap R^w L^p(M)))\, \Omega_0$ is a reducing subspace for $L^p(N)$. Since $(L^p(N) \cap R^w(L^p(M)))\, \Omega_0$ is essentially self-adjoint for $L^p(N)$, $P_{L^p(N)} \in L^p(N)'_w, P_{L^p(N)} \widehat{D}(L^p(N)) \subset \widehat{D}(L^p(N))$.

Now since $\overline{X\eta}(L^p(N)'_w)'$, for each $X \in L^p(N)$, we have $L^p(N)\Omega_0 = \overline{(L^p(N)'_w)'\Omega_0}$ that is $P_N = P((L^p(N)'_w)')$.

Let $S_{\Omega_0}$ and $S''_{\Omega_0}$ be the closures of the maps

$$S_{\Omega_0}A\Omega_0 = A^{\dagger}\Omega, A \in L^p(N)$$

$$. S''_{\Omega_0}B\Omega_0 = B^{*}\Omega, B \in (L^p(N)'_w)'$$

And so, $S_{\Omega_0} \subset S''_{\Omega_0}$ and $S_{\Omega_0} \neq S''_{\Omega_0}$ in general.

But then

$\Delta''^{it}_{\Omega_0}(L^p(N)'_w)'\,\Delta''^{-it}_{\Omega_0} = (L^p(N)'_w)', \forall t \in R$ where $\Delta''_{\Omega_0}$ is the modular operator for the full Hilbert Algebra $(L^p(M)'_w)'$.

This implies

$$P((L^p(N)'_w)')S''_{\Omega_0} \subset S''_{\Omega_0}P((L^p(N)'_w)')$$

And there exists a Conditional Expectation $E''_N$ of the Partial Generalized von Neumann Algebra $((L^p(M)'_w)', \Omega_0)$ with respect to $(L^p(N)'_w)'$.

And so

$E'_{L^p(N)}(A^{\dagger})\Omega_0 = P_{L^p(N)}A^{\dagger}\Omega_0 = P_{L^p(N)}S_{\Omega_0}A\Omega_0 = P_{L^p(N)}S''_{\Omega_0}A\Omega_0 = S''_{\Omega_0}P_{L^p(N)}A\Omega_0 = S''_{\Omega_0}E_{L^p(N)}(A)\Omega_0 = S_{\Omega_0}E_{L^p(N)}(A)\Omega_0 = E_{L^p(N)}(A)^{\dagger}\Omega_0$, for each $A \in L^p(M)$ which implies by the separateness of $\Omega$ that $E_{L^p(N)}$ is hermitian.

It is clear that $E(X) = X, \forall X \in L^p(N)$.

Now take arbitrary $A \in L^p(M)$ and $X \in L^p(N) \cap R^w(L^p(M))$.

Since $E_{L^p(N)}$ is hermitian, it follows that $A\Box X \in L^p(M)$ and $X \in L^p(N) \cap R^w(L^p(M))$.

Obviously,

$$\omega_{\Omega_0}(E_{L^p(N)}(A)) = \omega_{\Omega_0}(A), \text{ for all } A \in L^p(M).$$

Therefore $E_{NL^p(N)}$ is a Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$. □

**Theorem 7.3** Let $L^p(N) \subseteq L^p(M)$ be a Partial Generalized von Neumann Algebra associated with $L^p$-spaces on $D$ with a strongly cyclic and separating vector $\Omega_0 \in D$ satisfying the following conditions

i. $\overline{L^p(N)\Omega_0} = H_{L^p(N)}$
ii. $L^p(N)'_w \widehat{D}(L^p(N)) \subset \widehat{D}(L^p(N))$
iii. $\overline{L^p(N)\Omega_0}$ is essentially self-adjoint for $L^p(N)$.
iv. $\Delta''^{it}_{\Omega_0}(L^p(N)'_w)'\, \Delta''^{-it}_{\Omega_0} = (L^p(N)'_w)', \forall t \in R$ where $\Delta''_{\Omega_0}$ is the modular operator for the full Hilbert Algebra $(L^p(M)'_w)'$.

Then there exists a Conditional Expectation of $(L^p(M), \Omega_0)$ with respect to $L^p(N)$ if and only if $P_{L^p(N)}L^p(M)\Omega_0 = L^p(N)\Omega_0$.

**Proof.**

Since $L^p(N)\Omega_0 = \overline{L^p(N)\Omega_0}^{t_{L^p(N)}} = P_{L^p(N)}D$. It follows that $\overline{E_{L^p(N)}}(A) = \widehat{E_{L^p(N)}}(A)$ , for each $A \in L^p(M)$, and $\overline{E_{L^p(N)}}(A) \subset (L^p(N)_{P_{L^p(N)}})''_{wc}$.

□

## 8. Conclusion

In conclusion, this research successfully extends the mathematical framework for Conditional Expectations on $L^p$-spaces by transitioning from standard von Neumann Algebras to Partial Generalized von Neumann Algebras. By utilizing the modular automorphism group and modular operators for full Hilbert Algebras, the study establishes the necessary conditions for the existence and uniqueness of both weak and unbounded Conditional Expectations. The findings demonstrate that Unbounded Conditional Expectations represent a proper subspace of Weak Conditional Expectations. Ultimately, these extensions provide a more robust theoretical foundation for applications in Quantum Mechanics and other applied sciences where system elements are treated as operators acting within complex algebraic structures.

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